Area of a Surface of Revolution

CK-12

To access the online version of this FlexBook click the link below:

https://www.ck12.org/calculus/basic-formula-of-areas-of-surfaces-of-revolution-1501910835.49/lesson/Area-of-a-Surface-of-Revolution-CALC/

To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-source, collaborative, and web-based compilation model, CK-12 pioneers and promotes the creation and distribution of high-quality, adaptive online textbooks that can be mixed, modified and printed (i.e., the FlexBook® textbooks).

Printed: November 24, 2024 (PST)

AUTHORS
CK-12

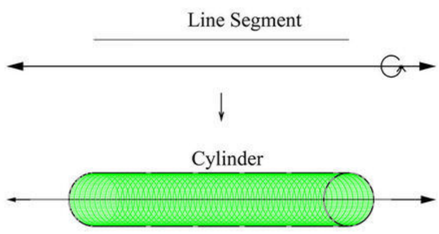
Definite integrals to find surface area of solids created by curves revolved around axes.

Area of a Surface of Revolution

Previous concepts developed methods for determining the volume of a solid of revolution, using cross-sectional areas (slices, disks, washers). Associated with the volume is the surface area generated by the length of the function curve as it is revolved around the axis-of-revolution. We know how to compute the differential and total arc length of a function curve. How do we formulate the rotation of this length about the axis-of-rotation to determine **area of a surface of revolution**?

Area of a Surface of Revolution

How do we find the area of a surface that is generated by revolving a curve about an axis or a line. For example, a circular cylinder can be generated by revolving a line segment about any axis that is parallel to it.



[Figure1]

The surface area, S, of that revolution can be fairly easily determined to be $S = 2\pi rh$, where r is the radius of revolution, and h is the length (height) of the line that is being revolved.

The following definition and formulation of the area of a surface of revolution is based on revolving a **differential arc length** about an axis and integrating over the length of the revolution.

The **area of a surface of revolution** is if $f(x)$ is a smooth and non-negative function in the interval $[a, b]$, then the surface area S generated by revolving the curve $y = f(x)$ about the x-axis is defined by

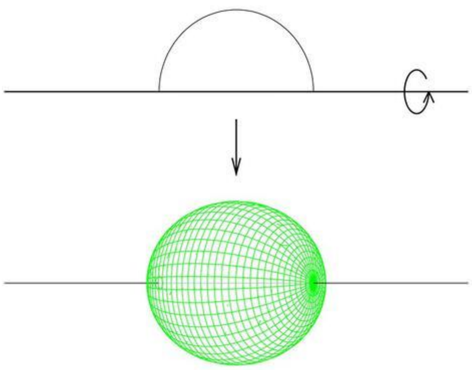
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Similarly: If $g(y)$ is a smooth and non-negative function in the interval $[c, d]$, then the surface area S generated by revolving the curve $x = g(y)$ about the y-axis is defined by

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\,dy = \int_c^d 2\pi g(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

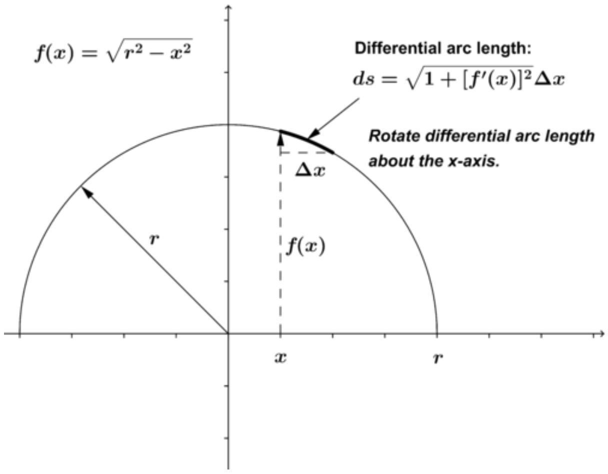
Note that in each integrand the terms $\sqrt{1 + [f'(x)]^2}\,dx$ and $\sqrt{1 + [g'(y)]^2}\,dy$ represent differential arc length along $f(x)$ and $g(y)$ introduced in a previous concept.

Show that the formula $S = 4\pi r^2$ for the surface area of a sphere with radius r, can be derived by rotating a semi-circle about the x-axis.



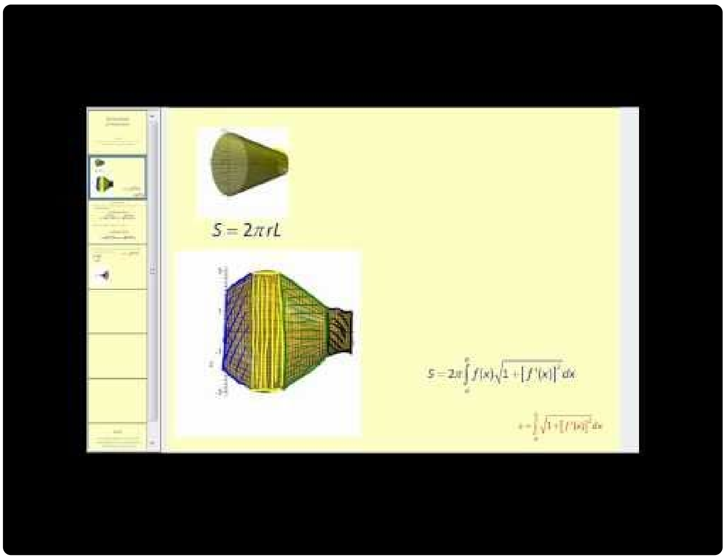
[Figure2]

The equation of a semi-circle in Quadrants I and II is given by $f(x) = \sqrt{r^2 - x^2}$. Because this function is symmetric about the y-axis, consider rotating only the Quadrant I portion of the semi-circle about the x-axis to generate half a sphere, then multiplying the area result by two.



[Figure3]

$$
\begin{aligned}
S &= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx \\
&= 2\int_0^r 2\pi\sqrt{r^2 - x^2}\sqrt{1 + \left[\frac{-x}{\sqrt{r^2 - x^2}}\right]^2}dx \\
&= 4\pi\int_0^r \sqrt{r^2 - x^2 + x^2}dx \\
&= 4\pi r[x]_0^r \\
&= 4\pi r^2
\end{aligned}
$$



https://www.ck12.org/flx/render/embeddedobject/81080

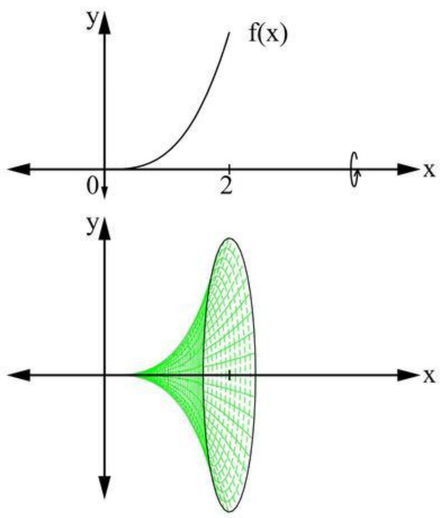
Examples

Example 1

Earlier, you were asked how to formulate the rotation of function curve length about the axis-of-rotation to determine area of a surface of revolution. This concept presented the formulation as the integral of the circular area segment $2\pi r \cdot ds$ along the appropriate coordinate axis.

Example 2

Find the surface area that is generated by revolving $y = x^3$ on [0, 2] about the x-axis.



[Figure4]

The surface area S is

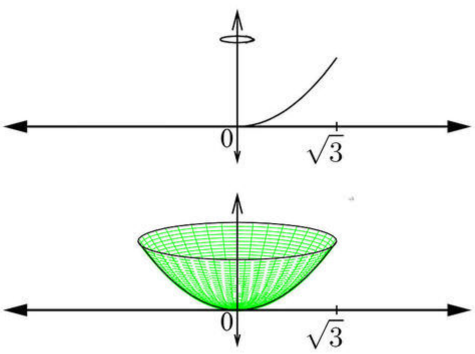
$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$
$$= 2\pi\int_0^2 x^3(1+9x^4)^{\frac{1}{2}}\,dx.$$

Using u-substitution by letting $u = 1+9x^4$ (and $du = 36x^3\,dx$)

$$S = 2\pi\int_1^{145} u^{\frac{1}{2}}\frac{du}{36}$$
$$= \frac{2\pi}{36}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{145}$$
$$= \frac{2\pi}{36}\cdot\frac{2}{3}[(145)^{\frac{3}{2}} - 1]$$
$$\approx \frac{4\pi}{108}[1745]$$
$$\approx 203$$

Example 3

Find the area of the surface generated by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{3}]$ about the y-axis.



[Figure5]

Since the curve is revolved about the y-axis, we apply

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

So we write $y = x^2$ as $x = \sqrt{y}$.

In addition, the interval on the x-axis $[0, \sqrt{3}]$ becomes [0, 3]. Thus

$$S = \int_0^3 2\pi\sqrt{y}\sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy.$$

Simplifying,

$$S = \pi \int_0^3 \sqrt{4y + 1}\, dy.$$

With the aid of u-substitution, let $u = 4y + 1$,

$$\begin{aligned} S &= \frac{\pi}{4} \int_1^{13} u^{\frac{1}{2}}\, du \\ &= \frac{\pi}{6}\left[(13)^{\frac{3}{2}} - 1\right] \\ &= \frac{\pi}{6}[46.88 - 1] \\ &\approx 24 \end{aligned}$$

Example 4

Find the surface area generated by revolving the function $f(x) = e^{0.1x}$ about the x-axis over the interval [0, 4].

Since the revolution is about the x-axis, the surface area formulation to use is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

With $f'(x) = 0.1e^{0.1x}$, the surface area is computed as follows:

$$\begin{aligned} S &= \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx \\ &= \int_0^4 2\pi e^{0.1x}\sqrt{1+[0.1e^{0.1x}]^2}dx \\ &= 2\pi\int_0^4 e^{0.1x}\sqrt{1+[0.1e^{0.1x}]^2}dx \\ &= 31.14 \qquad \ldots\text{Numberically evaluated using calculator.} \end{aligned}$$

The area of the surface of revolution is 31.14 square units.

Review

For #1-3 find the area of the surface generated by revolving the curve about the x-axis.

1. $y = 3x, 0 \le x \le 1$
2. $y = \sqrt{x}, 1 \le x \le 9$
3. $y = \sqrt{4 - x^2}, -1 \le x \le 1$

For #4-6 find the area of the surface generated by revolving the curve about the y-axis.

4. $x = 7y + 2, 0 \le y \le 3$
5. $x = y^3, 0 \le y \le 8$
6. $x = \sqrt{9 - y^2}, -2 \le y \le 2$
7. Show that the surface area of a sphere of radius r is $4\pi r^2$.
8. Show that the lateral area S of a right circular cone of height h and base radius r is $S = \pi r\sqrt{r^2 + h^2}$.
9. Find the surface area generated by rotating the line $y = x$ about the x-axis on the interval $0 < x < 5$.
10. Set up, but do not solve, an integral to calculate the surface area created by revolving $y = \cos x, \frac{\pi}{4} < x < \frac{\pi}{2}$ about the x-axis.

11. Find the surface area generated by rotating the curve $y = \sqrt{1 - x^2}, 0 < x < 0.5$ about the x-axis.
12. Find the surface area generated by rotating the curve $y = \sqrt{x}, 1 < x < 4$, about the x-axis.
13. Find the surface area generated by rotating the line $y = x$ about the y-axis on the interval $0 < x < 5$.
14. Set up, but do not solve, an integral to calculate the surface area created by revolving $y = \cos x, \frac{\pi}{4}, < x < \frac{\pi}{2}$ about the y-axis.
15. Find the surface area generated by rotating the curve $y = \sqrt{1 - x^2}, 0 < x < 0.5$, about the y-axis.
16. Set up, but do not solve, two integrals to calculate the surface area generated by rotating the curve $y = \sqrt{x}, 1 < x < 4$, about the y-axis. One should be in terms of x, one in terms of y.
17. Find the surface area generated by rotating the curve $y = x^2, 1 < x < 3$, about the y-axis.

Review (Answers)

Click HERE to see the answer key or go to the Table of Contents and click on the Answer Key under the 'Other Versions' option.

Vocabulary

Language: English ▾

Term	Definition
area of a surface of revolution	The area of a surface of revolution is the area created by a surface in Euclidean space created by rotating a curve around a straight line in its plane.
differential arc length	Differential arc length is the incremental arc length at a point along a function curve $f(x)$ (or $g(y)$) given by $\sqrt{1 + [f'(x)]^2}dx$ (or $\sqrt{1 + [g'(y)]^2}dy$).

1.0 REFERENCES

Image	Attributions
	**License:** CC BY-NC-SA