Derivatives and Integrals Involving Inverse Hyperbolic Functions

CK-12

To access the online version of this FlexBook
click the link below:

https://www.ck12.org/calculus/derivatives-and-integrals-of-inverse-hyperbolic-functions/lesson/Derivatives-and-Integrals-Involving-Inverse-Hyperbolic-Functions-CALC/

AUTHORS
CK-12

To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-source, collaborative, and web-based compilation model, CK-12 pioneers and promotes the creation and distribution of high-quality, adaptive online textbooks that can be mixed, modified and printed (i.e., the FlexBook® textbooks).

Copyright © 2025 CK-12 Foundation, www.ck12.org

The names "CK-12" and "CK12" and associated logos and the terms **"FlexBook®"** and **"FlexBook Platform®"** (collectively "CK-12 Marks") are trademarks and service marks of CK-12 Foundation and are protected by federal, state, and international laws.

Any form of reproduction of this book in any format or medium, in whole or in sections, must be attributed according to our attribution guidelines.
https://www.ck12info.org/about/attribution-guidelines

Except as otherwise noted, all CK-12 Content (including CK-12 Curriculum Material) is made available to Users in accordance with the CK-12 Curriculum Materials License
https://www.ck12info.org/curriculum-materials-license

CK-12 License

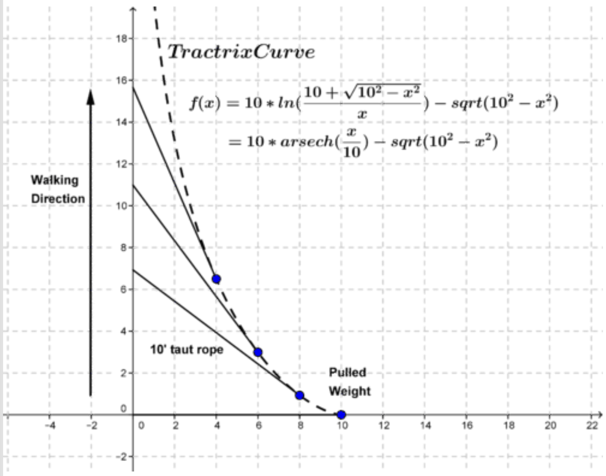
Complete terms for use for the CK-12 website can be found at:
http://www.ck12info.org/terms-of-use/

Printed: July 13, 2025 (PST)

Differentiation of the functions arsinh, arcosh, artanh, arscsh, arsech and arcoth, and solutions to integrals that involve these functions.

Derivatives and Integrals Involving Inverse Hyperbolic Functions

In the previous concept that introduced **inverse hyperbolic functions**, the concept question asked the name of the curve that described the path of a weight attached to a taut rope of a given length being pulled by someone in a direction perpendicular to the initial position of the weight. The curve is defined by the equation $y = a\,\text{sech}^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$, and is called a **tractrix**. It can also describe the path taken by the middle of the rear tire axle of a tractor trailer as it makes a turn perpendicular to its original direction of motion.



[Figure1]

Given the tractrix described in the previous concept by $y = 10\,\text{sech}^{-1}\left(\frac{x}{10}\right) - \sqrt{10^2 - x^2}$, can you determine the actual distance traveled by the weight along the curve between $x = 10\ ft$ (the start) and $x = 2\ ft$ (the end)?

Derivatives and Integrals with Inverse Hyperbolic Functions

Recall from the previous concepts that the hyperbolic functions and their inverses are defined as:

Table 1. Summary of Hyperbolic Functions and their Inverses

Hyperbolic Function	Inverse Hyperbolic Function	Domain	Range
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$-\infty \le x \le \infty$	$(-\infty, \infty)$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \ge 1$	$[0, \infty)$
$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\tanh^{-1} x = \dfrac{1}{2}\ln \dfrac{1+x}{1-x}$	$\lvert x\rvert < 1$	$(-\infty, \infty)$
$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$	$\operatorname{csch}^{-1} x = \sinh^{-1}\left(\dfrac{1}{x}\right)$ $= \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$x \ne 0$	$(-\infty, \infty)$
$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$	$\operatorname{sech}^{-1} x = \coth^{-1}\left(\dfrac{1}{x}\right)$ $= \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$	$0 < x \le 1$	
$\coth x = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$\coth^{-1} x = \tanh^{-1}\left(\dfrac{1}{x}\right)$ $= \dfrac{1}{2}\ln \dfrac{x+1}{x-1}$	$\lvert x\rvert > 1$	

The derivatives and integrals follow easily from the above.

Derivatives of the Inverse Hyperbolic Functions

Finding the **derivative** of each of the inverse hyperbolic functions is just a matter of differentiating each of the above expressions. If we let the argument of each inverse hyperbolic function be $u(x)$, then the generalized derivatives of the inverse hyperbolic functions are:

Let $u(x)$ be a differentiable function

Table 2. Derivatives of the Inverse Hyperbolic Functions

$\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \quad u \in \mathbb{R}$	$\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{\lvert u\rvert\sqrt{1+u^2}}\frac{du}{dx} \quad u \neq 0$
$\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \quad u > 1$	$\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx} \quad 0 < u < 1$
$\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \quad \lvert u\rvert < 1$	$\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx} \quad \lvert u\rvert > 1$

To see how easily the derivatives are derived, look at the following example.

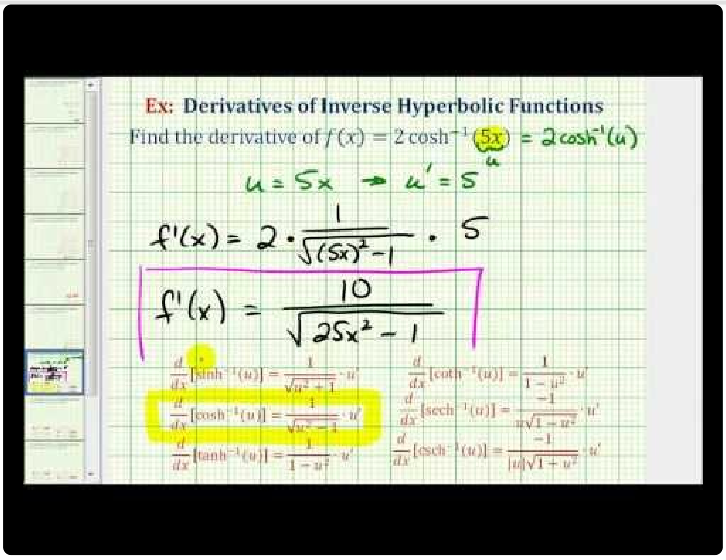
Derive the derivative of $\sinh^{-1} x$:

We know that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$. Let $u = x + \sqrt{x^2+1}$; then $\sinh^{-1} x = \ln(u)$.

Evaluating the derivative results in these steps:

$$\begin{aligned}
\frac{d}{dx}\sinh^{-1}\ x &= \frac{d}{dx}\ln(u) \\
&= \frac{1}{u}\frac{du}{dx} \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

Therefore, $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$



https://www.ck12.org/flx/render/embeddedobject/105322

Ex 2: Derivative of an Inverse Hyperbolic Fu
Mathispower4u

Watch on

Integrals Involving Inverse Hyperbolic Functions

Each of the derivative formulas presented above can be associated with an **integral** equation. For example,

$$\frac{d}{dx}[ar\sinh x] = \frac{1}{\sqrt{1+x^2}} \Leftrightarrow \int d[ar\sinh x] = \int \frac{1}{\sqrt{1+x^2}}dx = ar\sinh x + C.$$

Applying this procedure to the derivative of each inverse hyperbolic function results in these relationships:

Table 3. Integrals Involving Inverse Hyperbolic functions: Basic Form

$\int \frac{1}{\sqrt{1+x^2}}dx = \sinh^{-1} x + C$ $= \ln\left(x + \sqrt{x^2+1}\right) + C$	$\int \frac{1}{x\sqrt{1+x^2}}dx = -\text{csch}^{-1}x + C, \quad x \neq 0$
$\int \frac{1}{x^2-1}dx = \cosh^{-1} x + C, \quad x > 1$	$\int \frac{1}{x\sqrt{1-x^2}}dx = -\text{sech}^{-1}x + C, \quad 0 < \lvert x\rvert < 1$
$\int \frac{1}{1-x^2}dx = \tanh^{-1} x + C, \quad \lvert x\rvert < 1$	$\int \frac{1}{1-x^2}dx = \coth^{-1} x + C, \quad \lvert x\rvert > 1$

Notice that there are 5 different integrand forms. These integrand forms can all be generalized to provide a larger set of integrals that can be expressed as inverse hyperbolic functions. All of these generalized integrands can be setup so that using the u-substitution $u = \frac{b}{a}x$ transforms the integrand to one of the basic forms above. Three of the generalized integrands are shown below:

$$\frac{dx}{\sqrt{1+x^2}} \Rightarrow \frac{Adx}{\sqrt{a^2+b^2x^2}} = \frac{A}{a}\frac{dx}{\sqrt{1+\left(\frac{b}{a}x\right)^2}} = \frac{A}{a}\frac{a}{b}\frac{du}{\sqrt{1+u^2}} \qquad = \frac{A}{b}\cdot\frac{du}{\sqrt{1+u^2}}$$

$$\frac{dx}{1-x^2} \Rightarrow \frac{A}{a^2-b^2x^2}dx = \frac{A}{a^2}\frac{1}{1-\left(\frac{b}{a}x\right)^2}dx = \frac{A}{a^2}\cdot\frac{a}{b}\frac{du}{1-u^2} \qquad = \frac{A}{ab}\cdot\frac{du}{1-u^2}$$

$$\frac{dx}{|x|\sqrt{1+x^2}} \Rightarrow \frac{Adx}{|bx|\sqrt{a^2+b^2x^2}} = \frac{Adx}{\left|\frac{b}{a}x\right|a^2\sqrt{1+\left(\frac{b}{a}x\right)^2}} = \frac{Ab}{a^2}\frac{a}{b}\frac{du}{|u|\sqrt{1+u^2}} \qquad = \frac{A}{a}\cdot\frac{du}{|u|\sqrt{1+u^2}}$$

where $A, a,$ and b are constants. The other 2 forms can be generalized in a similar fashion.

The integrals of this generalized set of integrands can therefore also be expressed in terms of the inverse hyperbolic functions as presented in the table below:

$A, a,$

Substitution in Table I solution solves General Form integral.

Table 4. Integrals Involving the Inverse Hyperbolic Functions: General Form	
$\int \frac{Adx}{\sqrt{a^2+b^2x^2}} = \frac{A}{b}\sinh^{-1}\left(\frac{b}{a}x\right)+C$ $= \frac{A}{b}\ln\left(\frac{b}{a}x+\sqrt{\left(\frac{b}{a}\right)^2x^2+1}\right)+C$	$\int \frac{Adx}{x\sqrt{a^2+b^2x^2}} = -\frac{A}{a}\operatorname{csch}^{-1}\left\|\frac{b}{a}x\right\|+C$ $= -\frac{A}{a}\ln\left(\frac{1+\sqrt{1+\left(\frac{b}{a}\right)^2x^2}}{\left\|\frac{b}{a}x\right\|}\right)+C \qquad x\neq 0$
$\int \frac{Adx}{\sqrt{b^2x^2-a^2}} = \frac{A}{b}\cosh^{-1}\left(\frac{b}{a}x\right)+C$ $= \frac{A}{b}\ln\left(\frac{b}{a}x+\sqrt{\left(\frac{b}{a}\right)^2x^2+1}\right)+C$	$\int \frac{Adx}{x\sqrt{a^2-b^2x^2}} = -\frac{A}{a}\operatorname{sech}^{-1}\left\|\frac{b}{a}x\right\|+C$ $= -\frac{A}{a}\ln\left(\frac{1+\sqrt{1-\left(\frac{b}{a}\right)^2x^2}}{\left\|\frac{b}{a}x\right\|}\right)+C \quad 0<\|x\|<1$
$\int \frac{Adx}{a^2-b^2x^2} = \frac{A}{ab}\tanh^{-1}\left(\frac{b}{a}x\right)+C,\ \|x\|<a$ $= \frac{A}{2ab}\ln\left\|\frac{1+\frac{b}{a}x}{1-\frac{b}{a}x}\right\|+C, \quad \|x\|\neq\frac{a}{b}$	$\int \frac{Adx}{\sqrt{a^2-b^2x^2}} = \frac{A}{ab}\coth^{-1}\left(\frac{b}{a}x\right)+C,\ \|x\|>a$ $= \frac{A}{2ab}\ln\left\|\frac{1+\frac{b}{a}x}{1-\frac{b}{a}x}\right\|+C, \quad \|x\|\neq\frac{a}{b}$

Let's use the table above to evaluate the integral $\int_1^6 \frac{3}{\sqrt{9x^2-4}}dx$.

The integral is of the form $\int \frac{Adx}{\sqrt{b^2x^2-a^2}}$, which has the general solution $\frac{A}{b}\cosh^{-1}\left(\frac{b}{a}x\right)+C$.

Therefore $\int_1^6 \frac{3}{\sqrt{9x^2-4}}dx = \frac{3}{3}\left[\cosh^{-1}\left(\frac{3}{2}x\right)\right]_1^6$, and substituting the natural logarithm form yields

$$\begin{aligned}\int_1^6 \frac{3}{\sqrt{9x^2-4}}dx &= \ln\left(\frac{3}{2}6+\sqrt{\left(\frac{3}{2}\right)^2 6^2+1}\right)-\ln\left(\frac{3}{2}1+\sqrt{\left(\frac{3}{2}\right)^2 1^2+1}\right)\\ &= \ln(9+\sqrt{82})-\ln\left(\frac{3}{2}+\sqrt{\frac{13}{4}}\right)\\ &= \ln\left(\frac{9+\sqrt{82}}{\frac{3}{2}+\sqrt{\frac{13}{4}}}\right)\\ &= \ln\left(\frac{18.055}{3.303}\right)\\ &= 16.986 \approx 17.0\end{aligned}$$

Examples

Example 1

Earlier, you were asked to determine the distance traveled along the tractix defined by $y = 10\,\text{sech}^{-1}\left(\frac{x}{10}\right) - \sqrt{10^2 - x^2}$ between $x = 10\ ft$ (the start) and $x = 2\ ft$ (the end). T

Do you remember from an earlier concept how to compute the length of a curve, i.e., the arc length? The arc length L of the curve of a function $g(x)$ is computed from

$$Arc\ Length = L = \int_a^b \sqrt{1+f'(x)^2}\ dx.$$

Therefore,

$$L = \int_1^{10} \sqrt{1 + \left(\frac{d}{dx}\left[10\text{sech}^{-1}\left(\frac{x}{10}\right) - \sqrt{10^2 - x^2}\right]\right)^2} dx$$

$$= \int_1^{10} \sqrt{1 + \left(-\frac{\sqrt{10^2 - x^2}}{x}\right)^2} dx = 10 \ln x|_1^{10}.$$

The distance traveled along the curve is $L = 10 \cdot \ln 10 = 23\ ft$.

Example 2

Evaluate the derivative of the function $\tanh^{-1}(\cos 3x)$.

We know that $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$, so let $u = \cos 3x$.

Then,

$$\frac{d}{dx}\tanh^{-1}(\cos 3x) = \frac{1}{1-\cos^2 3x} \cdot \frac{d}{dx}\cos 3x$$
$$= -\frac{3}{\sin^2 3x} \cdot \sin 3x$$
$$= -\frac{3}{\sin 3x}$$

Example 3

Evaluate the derivative of $(\sinh^{-1} 2x)^{\frac{3}{2}}$.

$$\begin{aligned}\frac{d}{dx}(\sinh^{-1} 2x)^{\frac{3}{2}} &= \frac{3}{2}\sqrt{\sinh^{-1} 2x}\frac{d}{dx}(\sinh^{-1} 2x)\\ &= \frac{3}{2}\sqrt{\sinh^{-1} 2x}\frac{1}{\sqrt{1+4x^2}}\cdot 2\\ &= \frac{3\sqrt{\sinh^{-1} 2x}}{\sqrt{1+4x^2}}\\ &= \frac{3\sqrt{\ln\left(2x+\sqrt{1+4x^2}\right)}}{\sqrt{1+4x^2}}\end{aligned}$$

Example 4

Evaluate the integral $\int \frac{\cos x}{\sqrt{1+\sin^2 x}}dx$.

$$\begin{aligned}\int \frac{\cos x}{\sqrt{1+\sin^2 x}}dx &= \int \frac{1}{\sqrt{1+u^2}}du \qquad \ldots \text{Using the } u\text{-substitution:}\\ &\qquad\qquad u=\sin x, \text{ and } du = \cos x dx\\ &= \sinh^{-1} u + C\\ &= \sinh^{-1}(\sin x) + C\\ &= \ln\left(\sin x + \sqrt{1+\sin^2 x}\right) + C\end{aligned}$$

Review

For #1-7, evaluate the derivative.

1. $\sinh^{-1} 8x$
2. $\cosh^{-1} 7x^2$
3. $x(\tanh^{-1} 3x)^2$
4. $\cosh^{-1}(\csc x)$
5. $\tanh^{-1}(\cos x)$
6. $e^{-2x}\cosh^{-1}\ 3x$
7. $\operatorname{sech}^{-1}\left(\sqrt{3-x}\right)$

8. Show that $\int \frac{Adx}{\sqrt{b^2x^2-a^2}} = \frac{A}{b}\cosh^{-1}\left(\frac{b}{a}x\right)+C$, where $A, a,$ and b are constants.

For #9-15, evaluate the integral.

9. $\int \frac{4}{x\sqrt{1+9x^2}}dx$
10. $\int \frac{\cosh x}{\sqrt{\sinh^2 x-9}}dx$
11. $\int \frac{2x}{4-x^4}dx$ Hint: Let use u-substitution $u=x^2$.
12. $\int \frac{3}{(x+1)\sqrt{2x^2+4x+6}}dx$ Hint: Complete the square of the polynomial in the radical.
13. $\int \frac{1}{\sqrt{16x^2-25}}dx$
14. $\int \frac{e^{-x}}{4-e^{-2x}}dx$
15. $\int \frac{\cosh x}{\sinh^2 x+4\sinh x+2}dx$

Review (Answers)

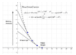
Click HERE to see the answer key or go to the Table of Contents and click on the Answer Key under the 'Other Versions' option.

Vocabulary

Language: English ▾

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y', $\frac{df}{dx}$ and \frac{df(x)}{dx}.
hyperbolic functions	The hyperbolic functions bear resemblance to the set of trigonometric functions. The basic hyperbolic functions are hyperbolic sine, $, and hyperbolic cosine,$\cosh.
integral	An integral is used to calculate the area under a curve or the area between two curves.
inverse	Inverse functions are functions that 'undo' each other. Formally: $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
tractrix	A tractrix is the name of the curve which is the solution to the differential equation $\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$, where a is a constant; $y = ar\text{sech}^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$.

1.0 REFERENCES

Image	Attributions
	**License:** CC BY-NC