Derivatives of Inverse Trigonometric Functions

CK-12

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Analyzing Graphs of Inverse Trigonometric Functions

Differentiation of arcsine, arccosine, arctangent, arccosecant, arcsecant, and arccotangent.

Derivatives of Inverse Trigonometric Functions

If you are interested in baseball, you know that making contact with a fast ball is difficult, especially when the ball is thrown at speeds that can be up to 100 mph. What makes it difficult? The difficulty is the batter's ability to follow the ball as it goes past the plate. The rate of change of the batter's observing angle as the ball goes by is just too fast compared with the eye/brain's ability to follow. The batter really does have to predict where the ball is going to be. Can you determine how fast a ball would have to be thrown so that a batter standing h feet from the plate would see the ball traveling with a rate of change of observing angle $\frac{d\theta}{dt}$ no greater than 3 radians/sec (about the maximum rate the eye/brain can follow)?

Derivatives of Inverse Trigonometric Functions

If $u(x)$ is a differentiable function of x then the generalized **derivative** formulas for the inverse trigonometric functions are:

Derivatives of Inverse Trigonometric functions

$\frac{d}{dx}[\arcsin u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\ \lvert u\rvert < 1$	$\frac{d}{dx}[\text{arc}\csc u] = \frac{-1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx},\ \lvert u\rvert > 1$
$\frac{d}{dx}[\arccos u] = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx},\ \lvert u\rvert < 1$	$\frac{d}{dx}[\text{arc}\sec u] = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx},\ \lvert u\rvert > 1$
$\frac{d}{dx}[\arctan u] = \frac{1}{1+u^2}\frac{du}{dx}$	$\frac{d}{dx}[\text{arc}\cot u] = \frac{-1}{1+u^2}\frac{du}{dx}$

Notice that there are just three results to remember because the other three are related by a change of sign:

- $\frac{d}{dx}[\arccos u] = -\frac{d}{dx}[\arcsin u]$
- $\frac{d}{dx}[\operatorname{arc}\cot u] = -\frac{d}{dx}[\arctan u]$
- $\frac{d}{dx}[\operatorname{arc}\csc u] = -\frac{d}{dx}[\operatorname{arc}\sec u]$

The above relationships can each be derived using the following type of procedure example for $\frac{d}{dx}[\arcsin u]$, where we let $y = \arcsin u$:

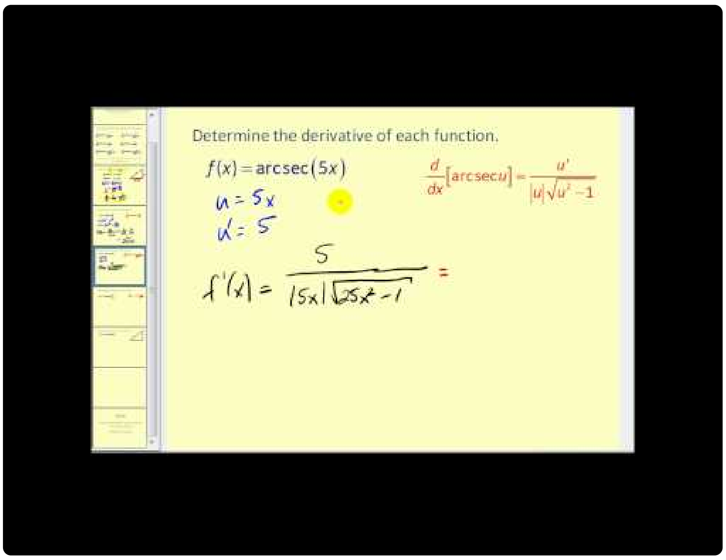
$$\begin{aligned} y &= \arcsin u \\ \sin y &= \sin(\arcsin u) \\ \sin y &= u \\ \frac{d}{dx}[\sin y] &= \frac{du}{dx} \\ \cos y \cdot \frac{dy}{dx} &= \frac{du}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y}\frac{du}{dx} \\ \frac{d}{dx}[\arcsin u] &= \frac{1}{\sqrt{1-\sin^2 y}}\frac{du}{dx} \\ \frac{d}{dx}[\arcsin u] &= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \end{aligned}$$

Now, use the formulas above to differentiate $y = \sin^{-1}\left(2x^4\right)$.

Let $u = 2x^4$. Then $y = \sin^{-1} u$.

Using the formula for $\frac{d}{dx}[\sin^{-1} u]$ gives:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \\ &= \frac{1}{\sqrt{1-\left(2x^4\right)^2}} \cdot \left(8x^3\right) \\ &= \frac{8x^3}{\sqrt{1-4x^8}} \end{aligned}$$



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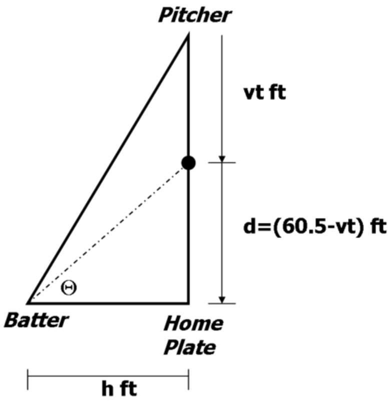
Examples

Example 1

Earlier, you were asked to determine how fast a ball would have to be thrown $(\text{speed} = v)$ so that a batter standing h feet from the plate would see the ball traveling with a rate of change of observing angle $\frac{d\theta}{dt}$ no greater than 3 radians/sec (about the maximum rate the eye/brain can follow).

A figure depicting the problem geometry is shown below.



[Figure1]

The batter's observing angle can be expressed as:

$$\tan\theta(t) = \frac{d}{h} = \frac{60.5 - vt}{h} \text{ or } \theta(t) = \arctan\left(\frac{60.5 - vt}{h}\right)$$

where 60.5 ft is the distance from the batter's box to the pitcher, v is the velocity of the ball, and t is time in seconds.

The rate of change of θ is
$\frac{d}{dt}\theta(t) = \frac{d}{dt}\arctan\left(\frac{60.5 - vt}{h}\right) = \frac{1}{1 + \left(\frac{60.5 - vt}{h}\right)^2}\left(\frac{-v}{h}\right)$.

When the ball arrives at the plate, $\frac{d}{dt}\theta(t) = \left(\frac{-v}{h}\right)$, so that if $h = 3\ ft$ and $\frac{d}{dt}\theta(t) = 3\ radians/sec$, then $v = 3 \cdot 3 = 9\ ft/sec \approx 6\ mph$. This is 15-17 times slower than the 90 to 100 mph speeds thrown by good pitchers.

Example 2

Differentiate $\tan^{-1}(e^{3x})$.

Let $u = e^{3x}$. Then $y = \tan^{-1} u$.

Using the formula for $\frac{d}{dx}[\tan^{-1} u]$ gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+u^2}\frac{du}{dx}\\ &= \frac{1}{1+\left(e^{3x}\right)^2}\cdot 3e^{3x}\\ &= \frac{3e^{3x}}{1+e^{6x}}\end{aligned}$$

Find $\frac{dy}{dx}$ if $y = \cos^{-1}(\sin x)$.

Let $u = \sin x$. Then $y = \cos^{-1} u$.

Using the formula for $\frac{d}{dx}[\cos^{-1} u]$ gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}\\ &= \frac{-1}{\sqrt{1-\sin^2 x}}\cos x\\ &= -1\end{aligned}$$

Example 3

Find the equation of the tangent line at $(1, 0)$ of the inverse function $\arctan(x+y) = y^2 + \frac{\pi}{4}$.

First we need to find the slope $y' = \frac{dy}{dx}$ at (1,0). Then we write the equation of the line at that point.

$$\begin{aligned}\frac{d}{dx}[\arctan(x+y)] &= \frac{d}{dx}\left[y^2+\frac{\pi}{4}\right]\\ \frac{1}{1+(x+y)^2}\frac{d}{dx}(x+y) &= 2yy'\\ \frac{1}{1+(x+y)^2}(1+y') &= 2yy'\\ y' &= \frac{1}{1-2y[1+(x+y)^2]}\end{aligned}$$

At (1,0), $y' = \frac{-1}{1-2\cdot 0[1+(1+0)^2]} = -1$.

The tangent line has the form $y = mx + b$:

$$\begin{aligned}&y = mx + b\\&0 = -1\cdot 1 + b\\&b = 1\end{aligned}$$

The tangent line is $y = -x + 1$.

Review

For #1-12, find the derivative with respect to the independent variable.

1. $y = \sec^{-1} x^2$.
2. $y = \frac{1}{\tan^{-1} x}$.
3. $y = \ln(\cos^{-1} x)$.
4. $y = \sin^{-1} e^{-4x}$.
5. $y = \sin^{-1}(x^2 \ln x)$.
6. $y = 3x^2 \arctan x$.
7. $y = \sin^{-1} \sqrt{2x}$.
8. $y = \tan^{-1} 2t^3$.
9. $x = \sin^{-1} \sqrt{1-t^4}$.
10. $y = t \cot^{-1}(1+3t^2)$.
11. $x = \frac{t}{\sqrt{1-t^2}} + \cos^{-1} t$.
12. $z = \cot^{-1}\left(\frac{y}{1-y^2}\right)$.
13. Find the relative extrema for $y = \arcsin x - 2\arctan x$.
14. What is the derivative of the function $\cos(\arcsin x)$.
15. Find the relative extrema for $z = \frac{-e^{-\arcsin x}}{\sqrt{1-x^2}}$.

Review (Answers)

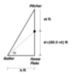
Click HERE to see the answer key or go to the Table of Contents and click on the Answer Key under the 'Other Versions' option.

Vocabulary

Language: English

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y', $\frac{df}{dx}$ and \frac{df(x)}{dx}.
inverse trigonometric [redacted]	An inverse trigonometric function is a function that reverses a trigonometric function, leaving the argument of the original trigonometric [redacted]

1.0 REFERENCES

Image	Attributions
	**License:** CC BY-NC