Inverse Hyperbolic Functions

CK-12

To access a customizable version of this book, as well as other interactive content, visit www.ck12.org

AUTHORS
CK-12

CK-12 Foundation is a non-profit organization with a mission to reduce the cost of textbook materials for the K-12 market both in the U.S. and worldwide. Using an open-source, collaborative, and web-based compilation model, CK-12 pioneers and promotes the creation and distribution of high-quality, adaptive online textbooks that can be mixed, modified and printed (i.e., the FlexBook® textbooks).

Printed: March 25, 2024 (PST)

Inverse Hyperbolic Functions

Domain, range, and basic properties of arsinh, arcosh, artanh, arcsch, arsech, and arcoth.

Inverse Hyperbolic Functions

Suppose you are standing, facing north, and holding the end of a 10 foot, taut rope attached to a heavy weight that is positioned directly east of you. As you walk north steadily pulling the heavy weight (without jerking) along the ground's, smooth, level surface, and keeping the rope taut, the path that the weight makes as it moves from its initial position is a curve that has a well known name in mathematics. This curve also describes the path taken by the middle of the rear wheel axle of a tractor trailer as it makes a turn perpendicular to its original direction of motion. Do you know the name of the curve? Do you have any idea how it relates to **inverse hyperbolic functions**?

Inverse Hyperbolic Functions

The inverse hyperbolic functions are denoted as follows:

Hyperbolic function/inverse pairs

$\sinh x \leftrightarrow \sinh^{-1} x \equiv ar\sinh x$	$\operatorname{csch} x \leftrightarrow \operatorname{csch}^{-1} x \equiv ar \operatorname{csch} x$
$\cosh x \leftrightarrow \cosh^{-1} x \equiv ar\cosh x$	$\operatorname{sech} x \leftrightarrow \operatorname{sech}^{-1} x \equiv ar \operatorname{sech} x$
$\tanh x \leftrightarrow \tanh^{-1} x \equiv ar\tanh x$	$\coth x \leftrightarrow \coth^{-1} x \equiv ar \coth x$

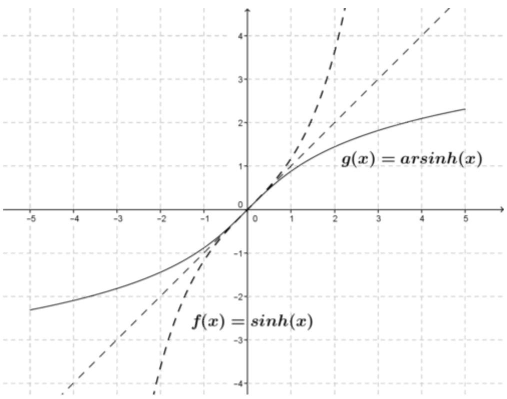
The notation that uses "ar" as a prefix is an alternative way of indicating the inverse function.

A function has an inverse if it is **one-to-one**. All of the hyperbolic functions except for $\cosh x$ are one-to-one functions and therefore have an inverse. Like the trigonometric functions, an inverse can be defined for $\cosh x$ by restricting its domain so that it is one-to-one. The table below provides a brief summary of characteristics of the six inverse hyperbolic functions.

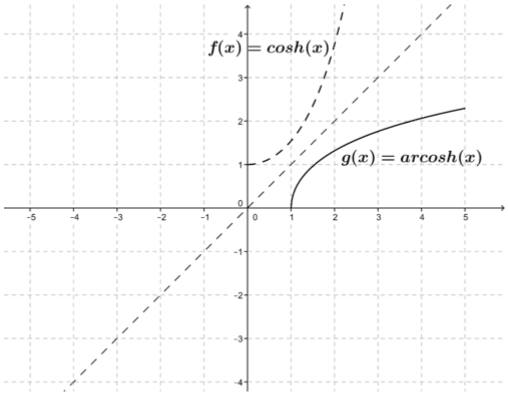
Inverse Function	Domain	Range	Basic Properties
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$-\infty \le x \le \infty$	$(-\infty, \infty)$	$\sinh^{-1}(\sinh x) = \sinh(\sinh^{-1} x) = x$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \ge 1$	$[0, \infty)$	$\cosh^{-1}(\cosh x) = \cosh(\cosh^{-1} x) = x$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$\lvert x\rvert < 1$	$(-\infty, \infty)$	$\tanh^{-1}(\tanh x) = \tanh(\tanh^{-1} x) = x$
$\operatorname{csch}^{-1}(x) = \sinh^{-1}\left(\frac{1}{x}\right)$ $= \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$x \ne 0$	$(-\infty, \infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = \operatorname{csch}(\operatorname{csch}^{-1} x) = x$
$\operatorname{sech}^{-1}(x) = \cosh^{-1}\left(\frac{1}{x}\right)$ $= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$0 < x \le 1$	$[0, \infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = \operatorname{sech}(\operatorname{sech}^{-1} x) = x$
$\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right)$ $= \frac{1}{2}\ln\frac{x+1}{x-1}$	$\lvert x\rvert > 1$	$y \ne 0$	$\coth^{-1}(\coth x) = \coth(\coth^{-1} x) = x$

The range is based on limiting the domain of the original function so that it is a one-to-one function.

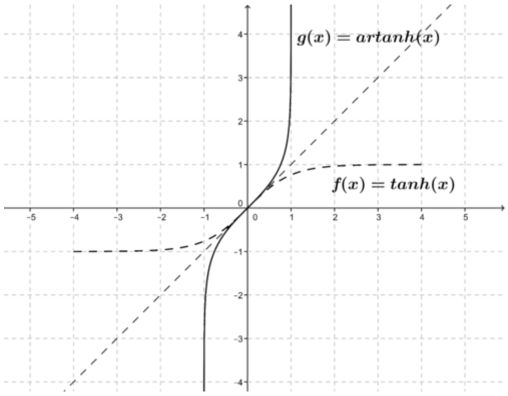
The hyperbolic function/inverse function pairs are graphed below:



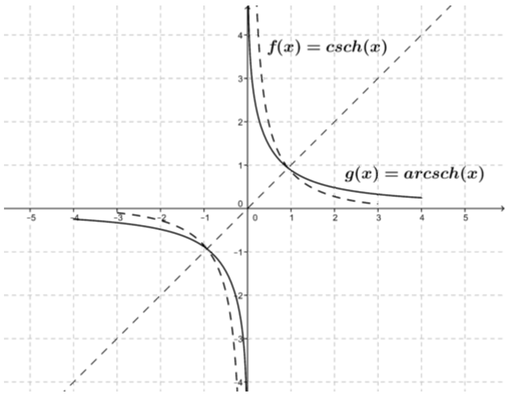
[Figure1]



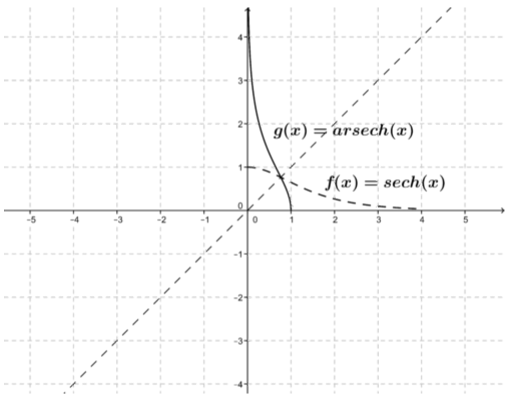
[Figure2]



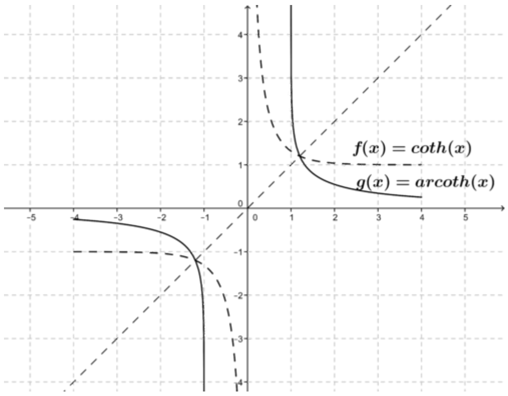
[Figure3]



[Figure4]



[Figure5]



[Figure6]

If you are wondering how the explicit expressions for the inverse hyperbolic functions are derived, take a look at this problem.

Show that $\sinh^{-1} x = ar\sinh x = \ln\left(x + \sqrt{x^2+1}\right)$.

Let $y = \sinh^{-1} x$, then by definition

$\sinh y = \dfrac{e^y - e^{-y}}{2} = \sinh(\sinh^{-1}\ x) = x$.

Also, $\cosh y = \dfrac{e^y + e^{-y}}{2} = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$, from the identity $\cosh^2 y - \sinh^2 y = 1$.

Consequently $\sinh y + \cosh y = e^y = x + \sqrt{1+x^2}$, which means

$$y = \ln\left(x + \sqrt{1+x^2}\right) = \sinh^{-1} x.$$

The other inverse hyperbolic functions can be derived in a similar manner.

Here are some examples of how to evaluate the hyperbolic inverse functions.

Evaluate each of the following:

$1. \tanh^{-1}\left(-\dfrac{7}{8}\right)$

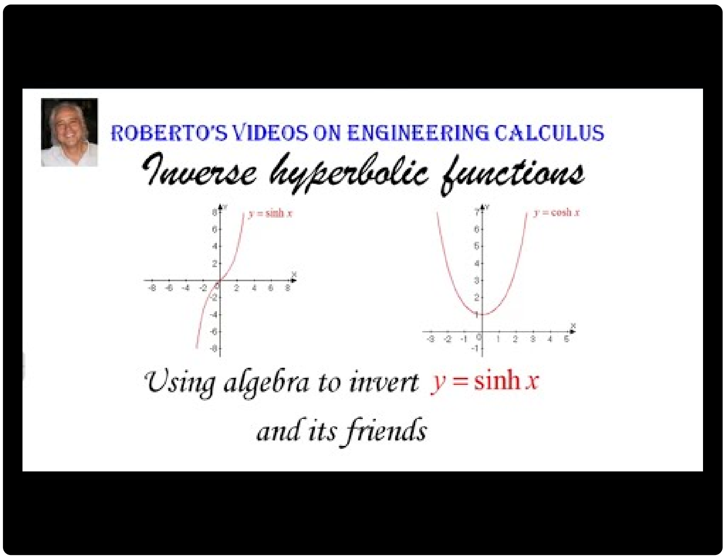
Given that $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$, then

$$\begin{aligned}\tanh^{-1}\left(-\frac{7}{8}\right) &= \frac{1}{2}\ln\left|\frac{1-\frac{7}{8}}{1+\frac{7}{8}}\right| \\ &= \frac{1}{2}\ln\left|\frac{\frac{1}{8}}{\frac{15}{8}}\right| \\ &= \frac{1}{2}\ln\frac{1}{15} \\ &\approx -1.35\end{aligned}$$

$2.\ \sinh^{-1} 3$

Given that $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$, then

$$\begin{aligned}\sinh^{-1} 3 &= \ln\left(3+\sqrt{3^2+1}\right) \\ &= \ln\left(3+\sqrt{10}\right) \\ &\approx 1.82\end{aligned}$$



https://www.ck12.org/flx/render/embeddedobject/105320

Examples

Example 1

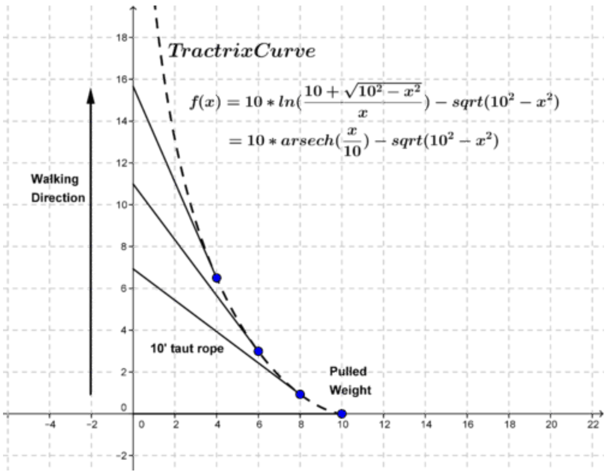
Earlier, you were asked about walking north steadily pulling a heavy weight (without jerking) along the smooth surface and keeping the rope taught. The weight starts directly east of you and the path that the weight makes as it moves from its initial position is a curve that has a well known name in mathematics. Do you know the name of the curve? Do you have any idea how it relates to inverse hyperbolic functions?

The curve is called a **tractrix,** expressed by the function:

$$y = a\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) - \sqrt{a^2-x^2}, \text{or } y = a r\operatorname{sech}^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}$$

$\operatorname{sech} x$ .whose first term is the inverse of the function

Here is a curve of this example.



[Figure7]

Example 2

Find the equation of the inverse of the function $y = \operatorname{sech}(2x-3)+4$ and identify any restrictions.

The function $y = \operatorname{sech}(2x-3)+4$ is not one-to-one. Using the restriction $x \geq \frac{3}{2}$ makes the function one-to-one, and enables definition of an inverse.

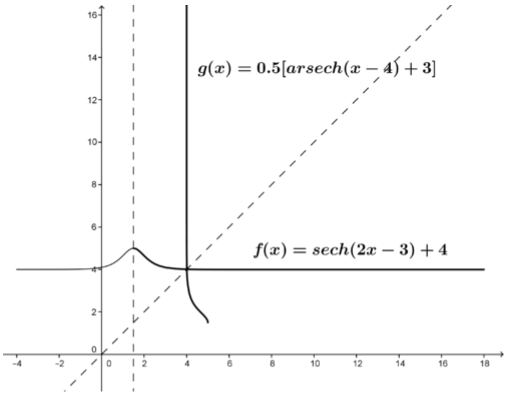
The inverse is obtained by solving the equation $x = \operatorname{sech}(2y-3)+4$ for y, as follows:

$$\begin{aligned}\operatorname{sech}(2y-3)+4&=x\\ \operatorname{sech}(2y-3)&=x-4\\ (2y-3)&=\operatorname{sech}^{-1}(x-4)\\ y&=\frac{1}{2}[\operatorname{sech}^{-1}(x-4)+3]\end{aligned}$$

Recall, however, that there is a restriction on the domain of the inverse $\operatorname{sech}^{-1}u$: $0<u\leq 1$.

Therefore $0<x-4\leq 1$, or $4<x\leq 5$.

The inverse of the function $y=\operatorname{sech}(2x-3)+4$ with $x\geq\frac{3}{2}$, is $y=\frac{1}{2}[\operatorname{sech}^{-1}(x-4)+3]$ with $4<x\leq 5$. The graph of function $f(x)$ and the inverse $g(x)$ are shown below.



[Figure8]

Example 3

A tsunami wave was observed to have a speed v of XX and a wavelength λ of YY. Give an estimate of the depth of the body of the water d using the following equation for the speed:

$v=\sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)}$, where

λ is the wavelength

d is the depth, and

g is the acceleration of gravity.

The speed is given by $v = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)}$.

Solving for the depth d gives

$\tanh\left(2\pi\frac{d}{\lambda}\right) = \frac{2\pi v^2}{g\lambda}$, so that

$d = \frac{\lambda}{2\pi}\tanh^{-1}\left(\frac{2\pi v^2}{g\lambda}\right)$.

With $\frac{2\pi v^2}{g\lambda} = \frac{2\pi \cdot v^2}{9.8 \cdot \lambda}$, then

$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$, and

$d = \frac{\lambda}{2\pi}\tanh^{-1}\left(\frac{2\pi v^2}{g\lambda}\right)$

Review

1. Show that $\sinh\left(\ln\left[x + \sqrt{1+x^2}\right]\right) = x$.

For #2-7, find the exact values of the inverse hyperbolic functions.

2. $\sinh^{-1} 0$
3. $\cosh^{-1} 0$
4. $\tanh^{-1} 0$
5. $\sinh^{-1} 2$
6. $\cosh^{-1} 1$
7. $\tanh^{-1} \frac{1}{2}$

For #8-12, find the exact value of x that satisfies the equation.

8. $\coth^{-1} x = \frac{1}{2}$
9. $\operatorname{csch}^{-1} 3x = \frac{1}{2}$
10. $\cosh^{-1} 4x = 1$.

11. $2\sinh x + \cosh x = 0$
12. $3\tanh x + 5\,\text{sech}\,x = 5$

For #13-15, find the equation of the inverse and identify any restrictions:

13. $y = 2 + \tanh\left(\frac{x+5}{3}\right)$
14. $y = \sinh(x^2)$
15. $y = \sinh^2(x-4) - 3$

Review (Answers)

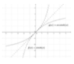
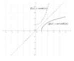
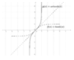
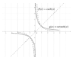
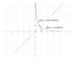
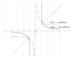
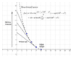
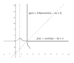
Click HERE to see the answer key or go to the Table of Contents and click on the Answer Key under the 'Other Versions' option.

Vocabulary

Language: English ▾

Term	Definition
hyperbolic functions	The hyperbolic functions bear resemblance to the set of trigonometric functions. The basic hyperbolic functions are hyperbolic sine, $, and hyperbolic cosine,$\cosh.
inverse	Inverse functions are functions that 'undo' each other. Formally: $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
One-to-one	A function is one-to-one if its inverse is also a function.
tractrix	A tractrix is the name of the curve which is the solution to the differential equation $\frac{dy}{dx} = -\frac{y}{\sqrt{a^2-y^2}}$, where a is a constant; $y = arsech^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}$.

1.0 REFERENCES

Image	Attributions
	**License:** CC BY-NC
	**License:** CC BY-NC
	**License:** CC BY-NC
	**License:** CC BY-NC
	**License:** CC BY-NC
	**License:** CC BY-NC
	**License:** CC BY-NC
	**License:** CC BY-NC