

Parabolas and the Distance Formula

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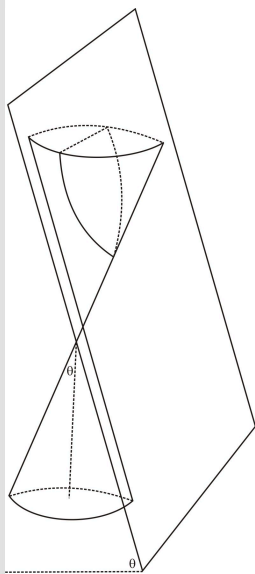


Quadratic as set of points
equidistant from a focus point
and directrix.

Parabolas and the Distance Formula

In previous lessons on **conic sections**, we discussed both the circle and the **ellipse**, which each result from "slicing" a cone clear through from left to right. In this lesson, we will discuss the shape formed when we slice through only one side of the cone, creating a bowl-shaped figure called a **parabola**.

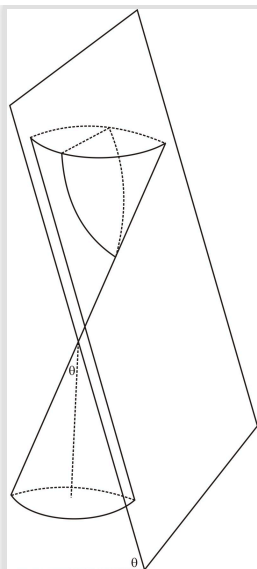
Consider the "hourglass" figure we used in the definitions of the circle and ellipse, created by connecting two infinite cones at their tips. What limitation would there be on the angle of the slice we would take out of one of the cones, if we wanted to only get a parabola (not get an ellipse, and not hit the other cone in any way)?



Parabolas and the Distance Formula

We've examined ellipses and circles, the two cases when a plane intersects only one side of the cone and creates a finite cross-section. Is it possible for a plane to intersect only one side of the cone, but create an infinite cross-section?

It turns out that this is possible if and only if the plane is parallel to one of the lines making up the cone. Or, in other words, the angle between the plane and the horizon, is equal to the angle formed by a side of the cone and the horizontal plane.

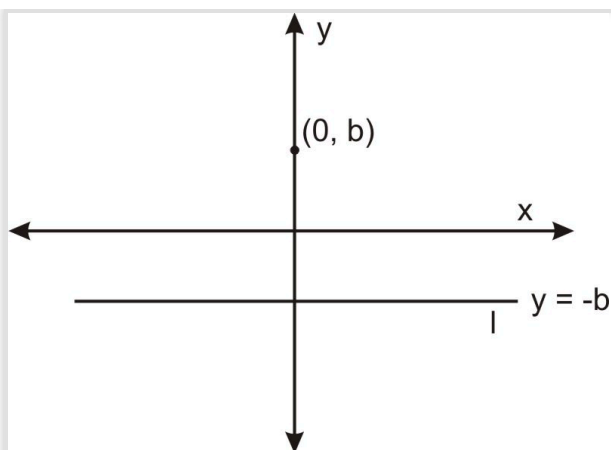


In the image above, if you tilt the plane a little bit to the left it will cut off a finite ellipse (possibly a very large one if you only tilt it a little.) Tilt the plane to the right and it will intersect both sides of the cone, making a two-part conic section called a hyperbola, which will be discussed in the next section.

When the plane is parallel to from the side of the cone, the infinite shape that results from the intersection of the plane and the cone is called a parabola. Like the ellipse, it has a number of interesting geometric properties.

The equation of a parabola is simpler than that of the ellipse. There are a couple of methods of deriving the equation of a parabola, in this lesson we explore the **distance formula**.

This first method arises directly from the **focus-directrix** property discussed in the previous section. Suppose we have a line and a point not on that line in a plane, and we want to find the equation of the set of points in the plane that is equidistant to these two objects. Without losing any generality, we can orient the line horizontally and the point on the **y**– axis, with the origin halfway between them. Since the parabola is the set of points equidistant from the line and the point, The parabola passes through the origin, $(0,0)$. The picture below shows this configuration. The point directly between the directrix and the focus (the origin in this case) is called the **vertex** of the parabola. Suppose the focus is located at $(0, b)$. Then the directrix must be $y = -b$.



Thus, the parabola is the set of points (x, y) equidistant from the line $y = -b$ and the focus point $(0, b)$. The distance to the line is the vertical segment from (x, y) down to $(0, -b)$, which has length $y - (-b) = y + b$. The distance from (x, y) to the focus $(0, b)$ is **distance** $= \sqrt{(x - 0)^2 + (y - b)^2}$ by the distance formula. So the equation of the parabola is the set of points where these two distances equal.

$$y + b = \sqrt{(x - 0)^2 + (y - b)^2}$$

Since distances are always positive, we can square both sides without losing any information, obtaining the following.

$$\begin{aligned} y^2 + 2by + b^2 &= x^2 + y^2 - 2by + b^2 \\ 2by &= x^2 - 2by \\ 4by &= x^2 \\ y &= \frac{1}{4b}x^2 \end{aligned}$$

But b was chosen arbitrarily and could have been any positive number, and for any positive number, a , there always exists a number b such that $a = \frac{1}{4b}$, so we can rewrite this equation as:

$$y = ax^2$$

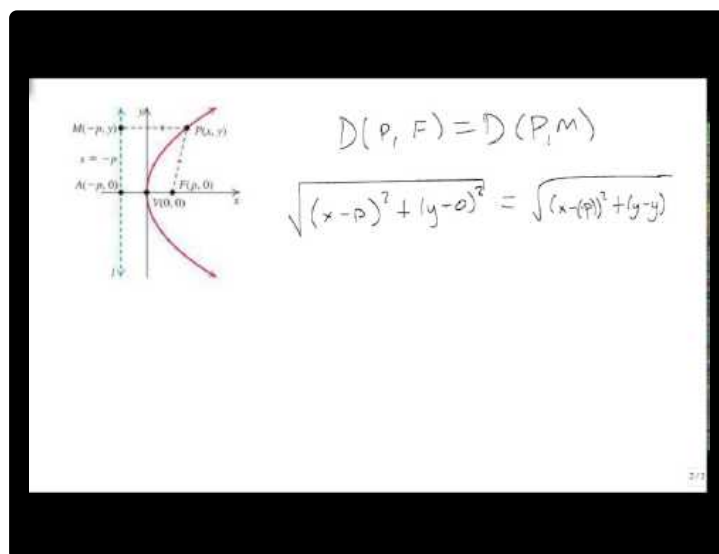
where a is any constant.

This is the general form of a parabola with a horizontal directrix, with a focus lying above it, and with a vertex at the origin. If a is negative, the parabola is reflected about the x -axis, resulting in a parabola with a horizontal directrix, with a focus lying below it, and with a vertex at the origin. The equation can be shifted horizontally or vertically by moving the vertex, resulting in the general form of a parabola with a horizontal directrix and passing through a vertex of (h, k) :

$$y - k = a(x - h)^2$$

Switching x and y , the equation for a parabola with a vertical directrix and with a vertex at (h, k) is:

$$x - h = a(y - k)^2$$



<https://www.ck12.org/flx/render/embeddedobject/187994>

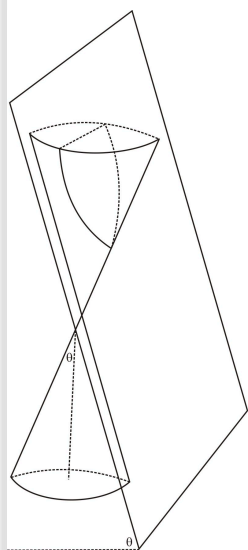
Examples

Example 1

Earlier, you were asked a question about the limitation on the angle of the slice taken out of one of the cones if you only wanted a parabola.

In order to only get a parabola cross section, the slice must be taken at the same angle as the side of the cone, that way the edge of the slice runs parallel to the edge of the cone and never intersects it at either top or bottom.

This can be seen by a close look at the image from above:



Example 2

Sketch a graph of the parabola $y = 3x^2 + 12x + 17$.

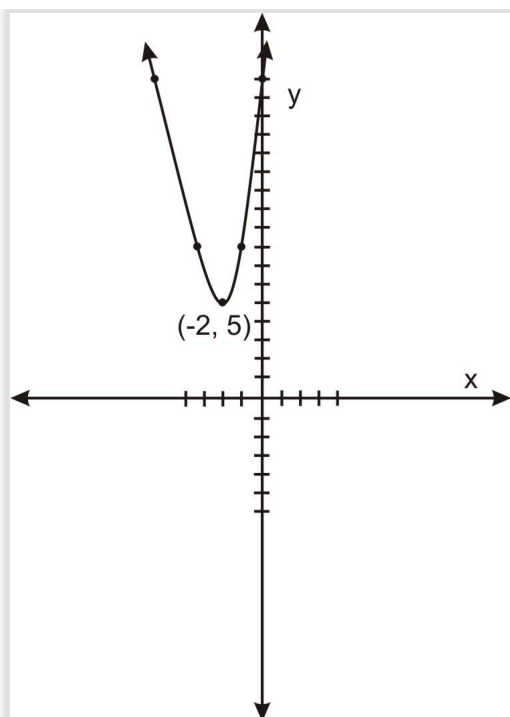
First, we need to factor out the coefficient of the x^2 term and complete the square:

$$\begin{aligned} y &= 3(x^2 + 4x) + 17 \\ y &= 3(x^2 + 4x + 4) + 17 - 12 \\ y &= 3(x + 2)^2 + 5 \end{aligned}$$

Now we write it in the form $y - k = a(x - h)^2$:

$$y - 5 = 3(x + 2)^2$$

So the vertex is at $(-2, 5)$ and plotting a few x -values on either side of $x = -2$, we can draw the following sketch of the parabola:



Example 3

Sketch a graph of the following parabola: $y = 2x^2 - 2x - 3$.

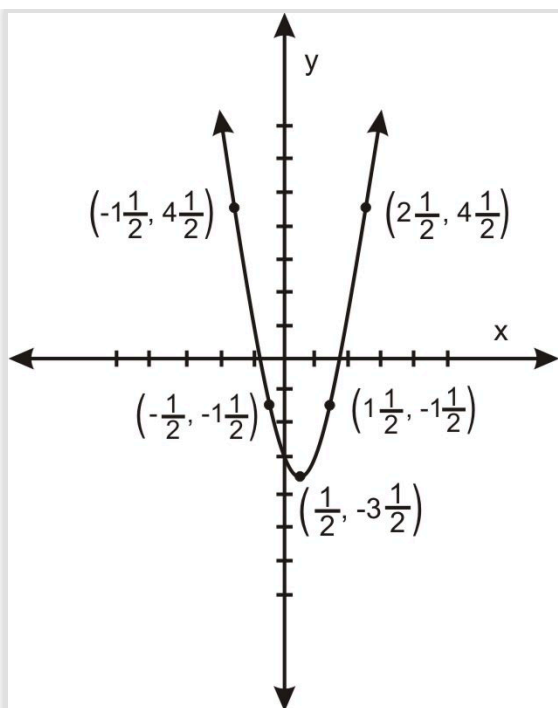
Factor out the 2: $y = 2(x^2 - x) - 3$

Complete the square: $y + \frac{1}{2} = 2(x^2 - x + \frac{1}{4}) - 3$

Add 3 to both sides and factor the trinomial: $y + 3\frac{1}{2} = 2(x - \frac{1}{2})^2$

The vertex (h, k) is: $(\frac{1}{2}, -3\frac{1}{2})$

Plot a couple of points to get:



Example 4

Sketch a graph of the following parabola: $3x^2 + 12x + 11 - y = 0$.

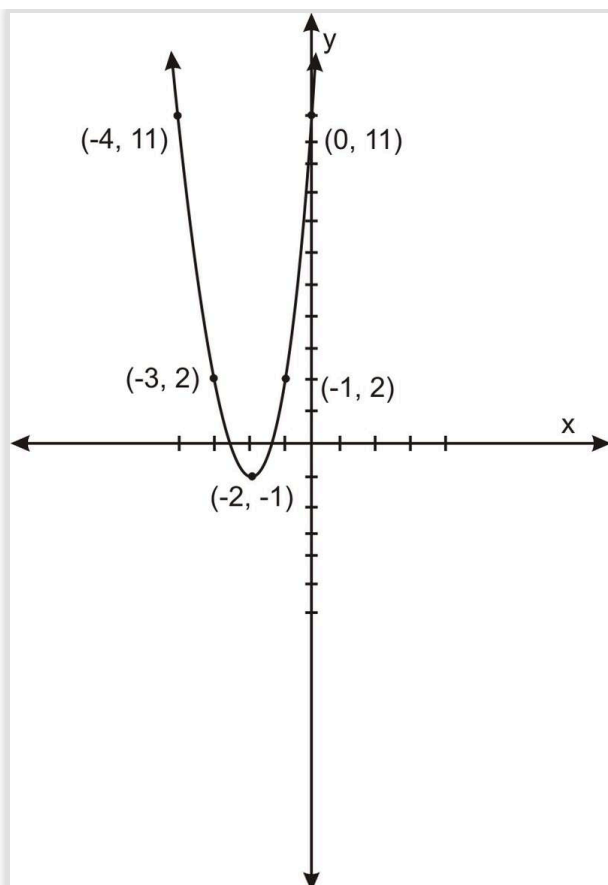
Factor out the 3 and move y and 11: $3(x^2 + 4x) = y - 11$

Complete the square: $3(x^2 + 4x + 4) = y - 11 + 12$

Factor the trinomial and collect like terms: $3(x + 2)^2 = y + 1$

The vertex (h, k) is at: $(-2, -1)$

Plot a couple of points to get:



Example 5

Identify which of the following equations are parabolas:

- a. $y - 5x + x^2 = 3$
- b. $x^2 - 3y^2 + 3y - 2x + 15 = 0$
- c. $x - 6y^2 + 20x - 100 = 0$

Recall that a parabola has a squared input term **only**:

Since b. $x^2 - 3y^2 + 3y - 2x + 15 = 0$ is squared on the input and **output** (in other words, both x and y are squared), it is not a parabola.

a. $y - 5x + x^2 = 3$ and c. $x - 6y^2 + 20x - 100 = 0$ are both parabolas.

By the way, after completing the square, equation a factors as:

$$(x - 1)^2 - 3\left(y - \frac{1}{2}\right)^2 + 14\frac{3}{4} = 0.$$

Example 6

Calculate the distance between (3, 4) and (9, 5).

To calculate the distance between (3, 4) and (9, 5), use the distance formula

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}:$$

$$\text{distance} = \sqrt{(9 - 3)^2 + (5 - 4)^2} \dots \text{Substitute}$$

$$\text{distance} = \sqrt{(6)^2 + (1)^2} \dots \text{Simplify}$$

$$\text{distance} = \sqrt{36 + 1} \rightarrow \sqrt{37}$$

$\therefore \sqrt{37}$ is the distance between (3, 4) and (9, 5).

Example 7

Calculate the distance between (-2, 7) and (11, 23).

To calculate the distance between (-2, 7) and (11, 23), use the distance formula

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}:$$

$$\text{distance} = \sqrt{(11 - (-2))^2 + (23 - 7)^2} \dots \text{Substitute}$$

$$\text{distance} = \sqrt{(13)^2 + (16)^2} \dots \text{Simplify}$$

$$\text{distance} = \sqrt{169 + 256} \rightarrow \sqrt{425} \rightarrow 5\sqrt{17}$$

$\therefore 5\sqrt{17}$ is the distance between (-2, 7) and (11, 23).

Convert from general to standard form of a

Jon Anderson



Watch on

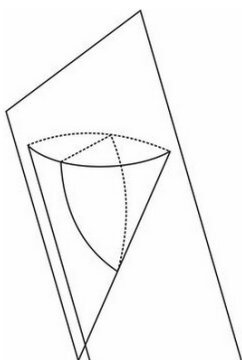
Review

Graph the following:

1. $y = -3(x - 1)^2 + 2$
2. $y = 3(x - 1)^2 - 1$
3. $x = -2(y + 2)^2 - 1$
4. $y = 3(x + 4)^2 - 1$
5. $y = (x - 3)^2$
6. $x = -3(y)^2 + 1$
7. $x = -(y - 3)^2 - 4$
8. $x = -3(y - 4)^2 + 4$
9. $y = 2(x + 3)^2 - 3$
10. $x = (y - 1)^2 - 2$

For questions 11-20, imagine a limited cone (not infinitely tall), as pictured below.

Assume the two coordinates listed represent the intersection of a parabolic curve and the top of the cone. If the top surface of the cone were represented by the x-axis, then the two coordinates could be considered the x-intercepts of the equation of the parabola. Find the distance between the points, and where required, the coordinates of the points.



11. Coordinates: (-20, -17) and (6, -1)
12. Coordinates: (-1, -5) and (6, -13)
13. Coordinates: (1, 2) and (5, -5)
14. Coordinates: (13, 12) and (15, 6)
15. Coordinates: (3, 9) and (6, -14)

16. $-25x^2 + 15x + 10 = 0$

17. $-24x^2 + 22x - 4 = 0$

18. $4x^2 - 24x + 32 = 0$

19. $24x^2 + 54x + 27 = 0$

20. $12x^2 + 25x + 12 = 0$

Review (Answers)

Click [HERE](#) to see the answer key or go to the Table of Contents and click on the Answer Key under the 'Other Versions' option.

Vocabulary

Language: **English** ▼

Term	Definition
Congruent	Congruent figures are identical in size, shape and measure.
directrix	The directrix of a parabola is the line that the parabola seems to curve away from. All points on a parabola are equidistant from the focus of the parabola and the directrix of the parabola.
Distance Formula	The distance between two points and can be defined as
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
focus	The focus of a parabola is the point that "anchors" a parabola. Any point on the parabola is exactly the same distance from the focus as from the directrix.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "focus", and a line on the exterior, called the "directrix". The directrix is vertical or horizontal, depending on the orientation of the parabola.
Vertex	The vertex of a parabola is the highest or lowest point on the graph of a parabola. The vertex is the maximum point of a parabola that opens downward and the minimum point of a parabola that opens upward.

1.0 REFERENCES

Image	Attributions
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